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On the Weak-Field Approximation in the Two-Body Problem of General Relativity

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Abstract

We consider the two-body problem of general relativity taking into account the retardation of interactions. The equations of motion are shown to be simplified, and this enables one to neglect the effects of heredity. The problem of validity of the approximations involved is considered in the extended-particle formalism. It is shown that under the correct treatment the self-interaction terms do not lead to unphysical solutions.

1. *Introduction*

The equations of motion of general relativity in the weak-field (or 'fast-motion') approximation were dealt with by many authors (Bennewitz & Westpfahl, 1971; Bertotti & Plebanski, 1960; Kerr, 1959; Kuhnel, 1964; Havas & Goldberg, 1962; Stephani, 1964). In these papers the solutions of the field equations are represented as the expansion in powers of the gravitational constant under the condition that the field is weak. It should be noted that such a consideration is not necessarily restricted to the case when the motion is a straight-line uniform one (Havas & Goldberg, 1962; cf. Infeld & Plebanski, 1960). It is known that if we consider the integrability conditions of the Einstein equations in the linear approximation as the exact ones, the uniform motion will result. However, we should not confine ourselves to the non-accelerated motion at the first step of the approximation, if we take into account that the conservation identities should be satisfied approximately (Havas & Goldberg, 1962). The explanation of this fact lies in the incorrect nature of the problem of finding the *approximate* equations of motion: small deviations of the metric in the integrability conditions may result in significant changes of the motion considered on sufficiently large intervals of time. In order to obtain satisfactory equations of motion one must use sufficiently high approximations of the metric, e.g. approximations of the same order (not the previous approximations) as that of the equations of motion. This leads to a rather complicated functional-differential structure of the equations: they contain the retarded arguments depending upon the unknown functions, the second-order equations contain the integro-differential 'tail' terms (Kuhnel, 1964; Bertotti & Plebanski, 1960; Stephani, 1964).

In the present paper we shall show that the equations of motion can be simplified in the weak field approximation. This enables one to neglect the effects of heredity in the following sense: after the transformations we obtain the ordinary differential equation and the terms depending upon the past history contribute only to the higher approximations. The transformations involve the expansion in powers of the retardation. This operation is not always rigorous, as the inspection of the self-action terms shows, e.g. it may lead to unphysical solutions analogous to 'runaway' solutions of the Dirac equation for point electron. It is shown that in the correct treatment these solutions do not arise.

We derive the equations of motion of the first approximation in the case of two extended bodies; the transition to the N-body problem is analogous. Remark are made concerning the structure of the second approximation. The effects of rotation are not considered.

2. *The Equations of Motion in the Weak-Field Approximation*

Consider the Einstein equations

$$G^{mn} = -kT^{mn} \tag{2.1}$$

where

$$T^{mn} = T_0^{mn} + S^{mn} \tag{2.2}$$

$$T_0^{mn}(x) = \int d^3y\,\rho(\bar{y})\,\delta(\bar{x} - \bar{Z}(x^0))\,\frac{dZ_y^m}{dx^0}\frac{dZ_y^n}{ds} \tag{2.2a}$$

here $x = (\bar{x}, x^0($ (we take $c = 1$), $m, n = 1, 2, 3, 0; Z_y^m = (\bar{Z}_y(x^0), Z_y^0(x^0)), Z_y^m$ is defined for $\bar{y} \in \text{supp}\,\rho(\bar{y})$ (supp $f(\bar{x})$ is the region where $f(\bar{x}) \neq 0$). S^{mn} describes the internal forces and is introduced to provide the stability with respect to the forces of mutual attraction of the elements of each particle. The expression for T_0^{mn} is chosen in analogy to the one for point particles (see Infeld & Plebanski, 1960), Z_y^n describing the world line of some element of one of the bodies. The expression (2.2a) is written in a fixed reference frame: this expression would change under coordinate transformations. We restrict our attention to the harmonic coordinates (see, e.g., Fock, 1964):

$$\frac{\partial}{\partial x^n}(\sqrt{(-g)}g^{mn}) = 0 \tag{2.3}$$

In these coordinates equation (2.1) may be written as

$$\Box h^{mn} = K^{mn} \tag{2.4}$$

where

$$K^{mn} = 2k[T^{mn} - \tfrac{1}{2} g^{mn}(T^{ab} g_{ab})] + 2\Gamma^{m,ab}\Gamma^{n}_{ab} - h^{ab} \frac{\partial^2 h^{mn}}{\partial x^a \, \partial x^b}$$

$h^{mn} = g^{mn} - \eta^{mn}$, $\eta^{mn} = \mathrm{diag}\,(1, -1, -1, -1)$.

The solution of (2.4) is sought for as the asymptotic expansion in powers of k:

$$_r h^{mn} = D_{\mathrm{ret}} * K^{mn}(_{r-1}h^{ab}) \tag{2.5}$$

$$_0 h^{mn} = 0$$

along with the solution of the equations of motion which follow from the conservation law:

$$T^{mn}{}_{;n} = 0 \tag{2.6}$$

In equation (2.5) $D_{\mathrm{ret}}(x) = (2\pi)^{-1}\theta(x^0)\,\delta[(x^0)^2 - (\bar{x})^2]$ is the retarded Green function of the wave equation, $\theta(x^0)$ being the Heavyside step function. We suppose that the convolution of D_{ret} with K^{mn} in (2.5) exists for all r involved. The subscript r indicates the order of iteration.

It is easy to see that in the first approximation the harmonicity condition (2.3) follows from (2.6) and, therefore, $_1h^{mn}$ gives the approximate solution of equation (2.1). The calculations in the higher approximations are more complicated. In this case, however, one can use the considerations analogous to those of Foures-Bruhat (1952) and Kerr (1959) in order to state the fulfilment of the harmonicity condition.

From (2.6), using (2.2a), we obtain the equations of motion for the elements of the bodies

$$\frac{d}{ds}\left(\frac{dZ^n_y}{ds}\right) + \Gamma^n_{ab}\frac{dZ^a_y}{ds}\frac{dZ^b_y}{ds} = -D_y S^n \tag{2.7}$$

where

$$S^n = S^{nm}{}_{;m}, \qquad D_y = \det\left\|\frac{\partial Z^i_y}{\partial y^j}\right\|, \qquad i, j = 1, 2, 3$$

We shall suppose that S^{mn} describes the internal forces acting inside the non-rotating (or slowly rotating) bodies (e.g. internal pressure). In accordance with our previous supposition we assume these forces provide sufficient rigidity for each body, so that $D_y = 1 + O(k)$ ($O(X)$ means the quantity of the same order as X). We also suppose that $S^{mn} \sim O(k) T^{mn}_0$. This is in agreement with the above assumption, because the condition of stability gives

$$S^n \sim (\Gamma^n_{ab})_{\text{self-action}} \frac{dZ^a_y}{ds}\frac{dZ^b_y}{ds} \sim O(k)$$

In the first approximation we have

$$ {}_1h^{mn}(x) = \frac{k}{\pi} \int d^3y \rho(y) \int dt\, \theta(x^0 - t)\, \delta[(x^0 - t)^2 - (\bar{x} - \bar{Z}_y)^2] $$

$$ \frac{\dot{Z}_y^m \dot{Z}_y^n - \frac{1}{2}(1 - \dot{\bar{Z}}_y^2)\eta^{mn}}{\sqrt{(1 - \dot{\bar{Z}}_y^2)}} $$

Here $Z_y^n = Z_y^n(t)$, the dot means differentiation with respect to the argument, $x = (\bar{x}, x^0)$.

Neglecting the higher-order terms in equation (2.7) one can obtain the three-dimensional equations of motion in the form (we do not write out the explicit form of the intermediate formulae)

$$ \frac{d}{dt} \frac{\dot{\bar{Z}}_y}{\sqrt{(1 - \dot{\bar{Z}}_y^2)}} - \frac{k}{4\pi} \int d^3y' \rho(\bar{y}') \int dt'\, \delta[t - t' - |\bar{Z}_y - \bar{Z}_{y'}|] $$

$$ \times \{\bar{F}[\bar{Z}_y, \bar{Z}_{y'}, \dot{\bar{Z}}_y, \dot{\bar{Z}}_{y'}] + \bar{g}[\bar{Z}_y, \bar{Z}_{y'}, \dot{\bar{Z}}_y, \dot{\bar{Z}}_{y'}, \ddot{\bar{Z}}_y, \ddot{\bar{Z}}_{y'}]\} = \bar{S}_y \quad (2.8) $$

Here $\bar{Z}_y = \bar{Z}_y(t), \bar{Z}_{y'} = \bar{Z}_{y'}(t'), \bar{S}_y$ are the spatial components of $S_y^n = (\bar{S}_y, S_y^0)$. Function $\bar{g}$ contains the terms proportional to $\ddot{\bar{Z}}_y$ and $\ddot{\bar{Z}}_{y'}$ and is bounded for $|\dot{\bar{Z}}_y| < c$ (c is some constant, $c < 1, y \in \text{supp}\, \rho$):

$$ |\bar{g}[\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{u}, \bar{v}]| \leqslant g_0(c) \frac{[|\bar{v}| + |\bar{u}|]}{|\bar{x} - \bar{y}|} \quad (2.9) $$

$$ |\bar{z}| \leqslant c < 1, \qquad |\bar{t}| \leqslant c, \qquad \bar{x} \neq \bar{y} $$

When the integration in (2.8) is performed, we obtain the functional-differential system containing the retardation $\tau_{yy'}(t)$, where

$$ \tau_{yy'}(t) = |\bar{Z}_y(t) - \bar{Z}_{y'}(t - \tau_{yy'}(t))| \quad (2.10) $$

Making use of the simple relations

$$ Z(t + u) = Z(t) + Z(t)u + \int_0^1 dv \int_0^v dv' Z(t + v'u)u^2 $$

$$ Z(t + u) = Z(t) + \int_0^1 dv\, Z(t + v\,.\,u)u $$

and the properties of equation (2.10), we can transform equation (2.8) as follows

$$\frac{d}{dt}\frac{\dot{\bar{Z}}_y}{\sqrt{(1-\dot{\bar{Z}}_y^2)}} - \frac{k}{4\pi}\int d^3y'\rho(\bar{y}')\{\bar{F}_1[\bar{Z}_y(t), \bar{Z}_{y'}(t), \dot{\bar{Z}}_y(t), \dot{\bar{Z}}_{y'}(t)] + \bar{g}_1[\bar{Z}_y, \bar{Z}_{y'}, \dot{\bar{Z}}_y, \dot{\bar{Z}}_{y'}, \ddot{\bar{Z}}_y, \ddot{\bar{Z}}_{y'}, t]\} = \bar{S}_y \quad (2.11)$$

Here $\bar{g}_1$ is the functional with the properties

$$|\bar{g}_1[\bar{Z}_y, \ldots, \ddot{\bar{Z}}_{y'}, t]| \leqslant \frac{g_1(c)[|\ddot{\bar{Z}}_y(t)| + ||\ddot{\bar{Z}}_{y'}||^t_{t-\tau_{yy'}}]}{|\bar{Z}_y(t) - \bar{Z}_{y'}(t)|}$$

We use the notation

$$||\bar{Z}||^{t_1}_{t_2} = \sup_{t\in[t_1, t_2]} |Z(t)|$$

Suppose that one can write

$$\bar{S}_y = {}_0S_y[\bar{Z}, \dot{\bar{Z}}] + {}_1\bar{S}_y[\bar{Z}, \dot{\bar{Z}}, \ddot{\bar{Z}}] \quad (2.12)$$

where ${}_0\bar{S}_y$ and ${}_1\bar{S}_y$ have the properties analogous to those of the second and the third term on the left-hand side of equation (2.11), respectively. From this it follows that ${}_1\bar{S}_y$ equals zero in the case of a uniform straight-line motion.

We shall also suppose that the internal forces described by S_y act only inside the bodies. In the case of two particles one can write

$$\rho(y) = \rho_a(y) + \rho_b(y), \qquad \text{supp } \rho_a \cap \text{supp } \rho_b = \emptyset$$

supp ρ_a, supp ρ_b are the connected sets. Here $a, b = 1, 2$ indicate the number of the particle.

In accordance with our supposition of stability we have (from the consideration of a uniform straight-line motion when the external forces acting upon the particle are absent)

$$\int d^3y'\rho_a(\bar{y}')\bar{F}_1[\bar{Z}_y(t), \bar{Z}_{y'}(t), \dot{\bar{Z}}_y(t), \dot{\bar{Z}}_{y'}(t)] - {}_0\bar{S}_y[\bar{Z}, \dot{\bar{Z}}] = 0$$

for $\bar{y} \in$ supp $\rho_a(\bar{y})$, $a = 1, 2$. Thus the self-action force in the second term of (2.11) is compensated of ${}_0\bar{S}_y$. The terms with the second derivatives contain the factor $\sim km_i/R_i \ll 1$ (we have assumed that ${}_0\bar{S}_y$ has the analogous structure), where

$$m_i = \int \rho_i(\bar{y})\, d^3y, \qquad i = 1, 2$$

R_i is the diameter of the ith particle, and it can be neglected after the self-action terms are removed.

Neglecting the terms proportional to the higher orders of R_i/r_{ab}, where r_{ab} is the distance between the particles, we have

$$\frac{8\pi}{km_b}\frac{d}{dt}\frac{\dot{\bar{Z}}_a}{\sqrt{(1-\dot{\bar{Z}}_a^2)}} = -\frac{(\bar{Z}_a-\bar{Z}_b)(1-\dot{\bar{Z}}_b^2)+((\bar{Z}_a-\bar{Z}_b)\dot{\bar{Z}}_b)\dot{\bar{Z}}_b}{[(\bar{Z}_a-\bar{Z}_b)^2(1-\dot{\bar{Z}}_b^2)+((\bar{Z}_a-\bar{Z}_b)\dot{\bar{Z}}_b)^2]^{3/2}}$$

$$\times\frac{2(1-\dot{\bar{Z}}_a\dot{\bar{Z}}_b)^2-(1-\dot{\bar{Z}}_a^2)(1-\dot{\bar{Z}}_b^2)}{\sqrt{(1-\dot{\bar{Z}}_a^2)}\sqrt{(1-\dot{\bar{Z}}_b^2)}}$$

$$+\frac{((\bar{Z}_a-\bar{Z}_b)\dot{\bar{Z}}_b)(1-\dot{\bar{Z}}_a\dot{\bar{Z}}_b)-(\dot{\bar{Z}}_a(\bar{Z}_a-\bar{Z}_b))(1-\dot{\bar{Z}}_b^2)}{[(\bar{Z}_a-\bar{Z}_b)^2(1-\dot{\bar{Z}}_b^2)+((\bar{Z}_a-\bar{Z}_b)\dot{\bar{Z}}_b)^2]^{3/2}(1-\dot{\bar{Z}}_a^2)^{3/2}(1-\dot{\bar{Z}}_b^2)^{1/2}}$$

$$\times[4(1-\dot{\bar{Z}}_a\dot{\bar{Z}}_b)(1-\dot{\bar{Z}}_a^2)\dot{\bar{Z}}_b-[2(1-\dot{\bar{Z}}_a\dot{\bar{Z}}_b)^2+(1-\dot{\bar{Z}}_a^2)(1-\dot{\bar{Z}}_b^2)]\dot{\bar{Z}}_a] \tag{2.13}$$

Here $\bar{Z}_i=\bar{Z}_i(t)$ is the three-dimensional trajectory of the ith particle. If $|\dot{\bar{Z}}_a|\ll 1$, $|\dot{\bar{Z}}_b|\ll 1$, equation (2.13) differs from the newtonian law of gravitation by the quantities of the second order in $|\dot{\bar{Z}}_a|$, $|\dot{\bar{Z}}_b|$.

As distinct from the first-approximation equations of Bertotti & Plebanski (1960), Kuhnel (1964), and Havas & Goldberg (1962), the equation obtained is an ordinary differential one. Equation (2.13) does not contain the retarded arguments, though the r retardation of interactions in our case is essential. We shall make some remarks concerning the second approximation. In this approximation we must take into account (a) nonlinear terms in the field equations, (b) the second-order terms on the right-hand side of equation (2.1), (c) the terms which were omitted in the first approximation in the equations of motion. Corrections (b) and (c) lead to the terms which depend on the behavior of the particle trajectories on some finite interval of time in the past. One can also use here the expansion with respect to retardation and obtain the ordinary differential equations for the trajectories. The concrete calculations are tied to the choice of S^{mn}. Corrections (a) give rise to the 'tail' terms (Bertotti & Plenbanski, 1960) depending upon all the past history of particles.

These corrections are due to the terms (see (2.4) and (2.5)):

$$D_{\text{ret}} * \left[2\,({}_1\Gamma^{m,ab}\,{}_1\Gamma^{n}_{ab}) - {}_1h^{ab}\frac{\partial^2\,{}_1h^{mn}}{\partial x^a\,\partial x^b}\right]$$

Using the expansion with respect to retardation and neglecting terms of the higher orders in k, one can easily obtain an expression in the form

$$\int d^3a\rho(\bar{a}) \int d^3b\,\rho(\bar{b}) \int d^3x' \int dt'\,\delta[\,t-t'-|\bar{x}-\bar{x}'|\,]$$

$$\times \frac{f[\bar{x}', \bar{Z}_a(t'), \bar{Z}_b(t'), \dot{\bar{Z}}_a(t'), \dot{\bar{Z}}_b(t')]}{|\bar{x}-\bar{x}'|\,|\bar{x}'-\bar{Z}_a(t')|^2\,|\bar{x}'-\bar{Z}_b(t')|^2}$$

$$\approx \int d^3a\rho(\bar{a}) \int d^3b\,\rho(\bar{b}) \int d^3x' \frac{f[\bar{x}', \bar{Z}_a(t), \bar{Z}_b(t), \dot{\bar{Z}}_a(t), \dot{\bar{Z}}_b(t)]}{|\bar{x}-\bar{x}'|\,|\bar{x}'-\bar{Z}_a(t)|^2\,|\bar{x}'-\bar{Z}_b(t)|^2}$$

The terms which are neglected in this approximation contain the second dervatives and contribute to the orders (of the equations of motion) higher than the second one. The expression obtained will not lead to the 'tail' terms in the equations of motion. This means that the 'tail' effects (i.e. the dependence upon all the past history) are undetectable in this approximation. The same is true of the dependence upon the past history which is due to the finite retardation. Note that analogous considerations are valid up to sufficiently high orders of the approximations involved, but not to the arbitrary orders.

The expressions obtained are well defined, but nothing has been said about the convergence of the integrals which are neglected in the course of approximation. Their convergence, however, does not affect the fact that the approximate solution obtained is the asymptotic solution of equation (2.1):

$$G^{mn}(h^{ab}) + kT^{mn}(h^{ab}) = \text{higher orders in } k$$

this can be verified explicitly.

One may suppose that this asymptotic solution gives a sufficiently good approximation to the exact solution. This statement is confirmed by the considerations of some simplified cases in the initial value problem which are similar to making estimates in the existence-uniqueness theorems (Choquet-Bruhat, 1970; Fisher & Marsden, 1972). However, these estimates appear to be uneffective in real applications. It is worthwhile to note the result of Fisher & Marsden (1973) which allows one to assert that the solution of linearised field equations is, in a certain sense, close to the exact one. However, their paper also does not contain the estimates which could be applied to the approximation methods in the problem of motion. The general problem of proving the approximations in general relativity still remains unsolved in spite of its importance (see Ryabushko, 1971).

3. *Some Aspects of the Approximations in the Equations of Motion*

The derivation of the equations of motion (2.13) involves the expansion with respect to retardation τ_{ab} (see (3.10)), in which the higher-order terms being neglected, that is an approximation like $x(t-r) \approx x(t) - r\dot{x}(t)$. It is well known (Elsgolts & Norkin, 1971) that such operation often gives an equation with properties different from those of the initial equation of motion. As we shall see later, taking account of this is important when we deal with the self-interaction terms.

Taking into account the condition of stability, we can write in the first approximation (rotation is neglected)

$$\bar{Z}_y(t) = \bar{Z}_i(t) + \bar{y}, \qquad \bar{y} \in \operatorname{supp} \rho_i \tag{3.1}$$

where $\bar{Z}_i(t)$ is the trajectory of some element of the ith body. We assume here that the point $\bar{y} = 0$ is situated inside the ith body (we consider one of the bodies). This means that the initial forces provide a sufficient rigidity for the particles. The solution of equation (2.10) is then estimated as

$$\tau_{iy}(t) \leqslant (1-c)^{-1} |\bar{y}|$$

if

$$|\dot{\bar{Z}}_i(t)| \leqslant c < 1, \qquad \bar{y} \in \operatorname{supp} \rho_i$$

Substituting (3.1) into (3.11) and taking into account the properties of the functional $\bar{g}_1$ from equation (2.11) and using equation (2.12), one can rewrite the equations of motion of the ith particle in the form

$$\frac{d}{dt} \frac{\dot{\bar{Z}}_i}{\sqrt{(1-\dot{\bar{Z}}_i^2)}} + \bar{G}(\dot{\bar{Z}}_i, \ddot{\bar{Z}}_i, t) = \bar{H}(t) \tag{3.2}$$

where $\bar{H}_i(t)$ describes the forces external with respect to the ith particle;

$$\bar{G}(\dot{\bar{Z}}_i, \ddot{\bar{Z}}_i, t) = k \int d^3y \rho_i(\bar{y}) \frac{\bar{h}(t, \dot{\bar{Z}}_i, \ddot{\bar{Z}}_i, \bar{y})}{|\bar{y}|} \tag{3.3}$$

the functional $h(t, \dot{\bar{Z}}_i, \ddot{\bar{Z}}_i, \bar{y})$ depending upon the behaviour of $\dot{\bar{Z}}_i, \ddot{\bar{Z}}_i$ on the segment $[t - \tau_{iy}, t]$. If $\bar{S}$ has the properties similar to those of the terms in (2.11), which are due to the gravitational forces, then $\bar{h}(t, \dot{\bar{Z}}_i, \ddot{\bar{Z}}_i, \bar{y})$ may be considered as a sufficiently smooth function of t and integrable function of $\bar{y}$, and for $|\bar{X}_1| \leqslant c < 1$

$$|\bar{h}(t, \bar{X}_1, \bar{X}_2, \bar{y})| \leqslant h_0 \|\bar{X}_2\|_{t-\tau_{0y}}^t \tag{3.4}$$

$\rho_i(\bar{y})$ is also supposed to be integrable.

Using the explicit form of equation (2.11), one can show that the operator $\bar{G}$ in (3.2) is Lipshitz-continuous in a certain sense, and proves the existence-uniqueness theorem in analogy to the corresponding theorem for equations of the neutral type (Driver, 1963).

If we transform the self-interaction terms using the expansion with respect to the retardation τ_{iy} in the integrand of (3.3) and perform the integration, we shall obtain the equation of motion containing the derivative of the particle acceleration with respect to time. The equation obtained would be analogous to that of Dirac (1938). The presence of the third derivatives of the particle trajectories creates some difficulties. In particular, in order to remove the unphysical solutions additional assumptions are needed (Rohrlich, 1961). We shall make a few remarks concerning the equations with the third derivative which is the consequence of the point structure of the particle. The third derivative arises in the calculation of the self-interaction force. One can mention two ways of derivation: (1) The self-interaction force is calculated for an extended body and taking the limit $r \to 0$ is used (r is the size of the particle). (2) The resulting expressions are obtained through the renormalisation procedure from the singular potentials.

Both kinds of derivations have a restricted range of validity (Ginzburg, 1946, 1969). Both the methods cannot be considered as quite rigorous ones because the operations used here could spoil the uniqueness of solutions and would lead to other troubles. The runaway solutions in this case are understandable from the following point of view: the energy necessary for such a motion is extracted from the infinite self-energy of the particle (Teitelboim, 1970). One should also take into account that the classical equations are invalid at small distances.

For these reasons it would be desirable to have the equation of motion of an extended particle. This problem was considered in electrodynamics by Fradkin (1950), Belousov (1939), Ginzburg (1946) and Kaup (1966). The most satisfactory solution of this problem is due to Kaup. His equation differs from that of Dirac (1938), and possesses good physical properties. However, the results of Kaup (1966) are not applicable to our case without making substantial modifications. In this connection we shall show that equation (3.2) does not admit the 'runaway' solutions by means of an alternative method.

Suppose that the distance between the particles increases to infinity so that the interaction vanishes: $\bar{H}_i(t) \to 0$. It is easy to prove that there exists a function $f(t)$ such that

$$f(t) \geqslant |\bar{H}_i(t)|, \qquad \underset{t\to\infty}{f(t) \to 0}$$

$$f_0 = \sup_{t\in[t_0,\infty)} \{f(t-d')[f(t)]^{-1}\} < \infty$$

Here $|\dot{\bar{Z}}_i| \leqslant c < 1$, $d' = (1-c)^{-1}d$, d being the diameter of the particle.

From (3.2)-(3.4) one obtains (we suppress the index i)

$$|\dot{\bar{U}}(t)| - a\|\dot{\bar{U}}\|^t_{t-d'} \leqslant |\bar{H}(t)| \tag{3.5}$$

where $\bar{U} = \dot{\bar{Z}}(1-\dot{\bar{Z}}^2)^{-1/2}$, $a = 2\pi k\rho_0 d^3 h_0$, $\rho_0 = \max \rho(\bar{y})$. Using (3.5) we obtain

$$\|\dot{\bar{U}}f^{-1}\|^\infty_{t_0} \leqslant (1-af_0)^{-1}(\|\bar{H}f^{-1}\|^\infty_{t_0} + aU_0) \tag{3.6}$$

where $U_0 = \|\dot{\bar{U}}\|^{t_0}_{t_0-d'}$ is defined by the initial data given on the segment $[t_0 - d', t_0]$. It was supposed here that $af_0 < 1$. This inequality is admissible because in the real case the maximal density is small: $\rho_0 \sim mR^{-3}$, $kmR^{-1} \ll 1$ (moreover, $af_0 \ll 1$). From (3.6) we infer that $\dot{\bar{U}} \to 0$ for $t \to \infty$, i.e. the equations of motion do not admit the unphysical solutions.

Using (3.6) it is easy to estimate the error of the approximation when we neglect the self-interaction terms in (2.11):

$$|\bar{G}(\dot{\bar{Z}}, \ddot{\bar{Z}}, t)| \lesssim O\left(\frac{km}{R}\right) f(t)$$

i.e. the self-interaction terms do not contribute to the first order of our approximation. This is in agreement with the result of Carmeli (1968). However, the self-interaction effects must be taken into account in the second approximation and it is desirable to give a more convenient expression for (3.3). The formula which could be obtained by expanding with respect to retardation leads to the 'runaway' solutions. However, it will be easy to obtain a more convenient equation if we use the expression for the second derivative from the first-order equation (2.13). The equation that can be obtained will not contain the derivatives of the order higher than the second. An analogous procedure can be applied to the higher approximations.

Thus far we have restricted our attention to the self-interaction terms. The expansion with respect to the retardation was used also in the terms describing the action of one particle on the other. This procedure can be justified, if the resulting expressions do not contain the derivatives of the order higher than the second. The estimates are analogous to the corresponding ones in the case of small velocities (Zhdanov & Pyragas, 1973) and we only note the main features of our reasoning. In this case the finite structure of the particles does not give rise to any difficulties and we can regard the particles as the point ones. The terms neglected in the approximation can be written as (cf. equation (2.9))

$$\frac{km\bar{F}'(\bar{Z}_a, \bar{Z}_b, \dot{\bar{Z}}_a, \dot{\bar{Z}}_b, \ddot{\bar{Z}}_a, \ddot{\bar{Z}}_b)}{|\bar{Z}_a(t) - \bar{Z}_b(t - \tau_{ab}(t))|}$$

where $\bar{F}'$ satisfies the following estimates

$$|\bar{F}'(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V}, \bar{W})| \leqslant F'_0(c)(|\bar{V}| + |\bar{W}|)$$

for

$$|\bar{Z}| \leqslant c < 1, \qquad |\bar{U}| \leqslant c$$

Using the above formulae one can estimate the error of approximation in analogy to (Zhdanov & Pyragas, 1973):

(1) for the characteristic time of a finite motion (e.g. the period of the revolution of a two-body system),

(2) for the whole time in the case of an infinite motion using the conditions of complete dispersion (their proof in our case is identical).

The result shows that the error is of the order $O(km/r)$, where r is the characteristic or initial interparticle distance.

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